Network-induced oscillatory behavior in material flow networks and irregular business cycles

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Network theory is rapidly changing our understanding of complex systems, but the relevance of topological features for the *dynamic* behavior of metabolic networks, food webs, production systems, information networks, or cascade failures of power grids remains to be explored. Based on a simple model of supply networks, we offer an interpretation of instabilities and oscillations observed in biological, ecological, economic, and engineering systems. We find that most supply networks display *damped oscillations*, even when their units—and linear chains of these units—behave in a *nonoscillatory* way. Moreover, networks of *damped* oscillators tend to produce *growing* oscillations. This surprising behavior offers, for example, a different interpretation of business cycles and of oscillating or pulsating processes. The network structure of material flows itself turns out to be a source of instability, and cyclical variations are an inherent feature of decentralized adjustments.

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I. INTRODUCTION

Network theory [1,2] is an important key to explaining complex systems. Topological features [3] determine interesting properties of socioeconomic networks [4] or the robustness of the World Wide Web [5], of sensory systems [6], and food webs [7]. Their relevance for dynamic features, however, is not well understood. Specific network structures appear to be responsible for the beating of a leech's heart [8], pumping effects in some slime molds [9], strong variations in ecosystems [10], or glycolytic oscillations in yeast cells [11]. Many of those systems, including molecular networks [12], can be viewed as particular supply networks. Their functioning is decisive for natural and man-made systems, but prone to oscillations: For example, industries often suffer from over-reactions in production [13,14], disaster management struggles with a temporary clustering of forces and materials at some places while they are missing at others [15], and national economies are characterized by "booms" and "recessions" [16,17]. Based on a simple model of interactive flows, we will show that the likely reason for these instabilities and oscillations is the underlying network structure of supply systems. This is illustrated with a model of macroscopic commodity flows.

The relevance of our approach for physics is twofold: First, we generalize conservation equations for flows in three-dimensional continuous spaces to network flows. Second, we show that chains of damped elements behave qualitatively different from networks of these elements, if interactions are not symmetric. This applies, for example, to some networks of electronic components.

II. MODEL OF MATERIAL FLOW NETWORKS

Our simple model of supply networks describes the generation or delivery of products, substances, materials, or other resources of kind i at a certain rate $Q_i(t) \geqslant 0$ as a function of time t. A share $X_{ij}(t) \geqslant 0$ of that output is required as input for generating or delivering resource j, and a share $Y_i(t) \geqslant 0$ is absorbed, for example by consumption, degeneration, or loss. If $\sum_j X_{ij}(t) + Y_i(t)$ does not add up to $Q_i(t)$, this is assumed to result in a corresponding change $[N_i(t+\Delta t) - N_i(t)]$ of the stock, inventory, or concentration $N_i(t)$ of resource i per unit time Δt. This assumption implies a *conservation equation* for resources (which can be generalized). In the limit $\Delta t \rightarrow 0$ it reads

$$\frac{dN_i(t)}{dt} = \underbrace{Q_i(t)}_{\text{supply}} - \underbrace{\left[\sum_{j=1}^{m} a_{ij} Q_j(t) + \overbrace{Y_i(t)}^{\text{outflow}}\right]}_{\text{demand}}, \tag{1}$$

considering that the flow $X_{ij}(t)$ of resources from supplier i represents a share $a_{ij}Q_j(t) \geqslant 0$ of the entire inputs used by supplier j. In Leontief's input-output model of macroeconomics [18], the m suppliers correspond to different economic sectors producing one kind of commodity i each. $Q_i(t)$ describes the output flow and $N_i(t)$ the inventory of commodity i. The stoichiometric coefficients $a_{ij} < 1$ define an input matrix $\mathbf{A} = (a_{ij})$ and reflect the technologically determined supply network between economic sectors i and j. For simplicity, we will treat the input coefficients a_{ij} as constants.

The change of the delivery rate $Q_i(t)$ in time is generally a function W_i of the stock levels or concentrations N_j, their temporal change dN_j/dt, and the delivery rates Q_j themselves: $dQ_i/dt = W_i(\{N_j\}, \{dN_j/dt\}, \{Q_j\})$ with $j \in \{1, \ldots, m\}$. In our model of macroeconomic output flows, the delivery rates $Q_i(t)$ are adjusted in response to two criteria: First, if the current inventory $N_i(t)$ exceeds some "optimal" level N_i^0, the delivery rate is reduced and vice versa. A certain inventory level is desireable to cope with variations in the demand. Second, if inventories are growing ($dN_i/dt > 0$), i.e., if the current supply $Q_i(t)$ exceeds the current demand $Y_i(t) + \sum_j a_{ij} Q_j(t)$, this is an independent reason to reduce the delivery rate $Q_i(t)$. The following equation for the relative change in the delivery rate expresses these responses and guarantees a non-negativity of $Q_i(t)$:

$$\frac{1}{Q_i(t)}\frac{dQ_i}{dt} = \hat{\nu}_i\left(\frac{N_i^0}{N_i(t)} - 1\right) - \frac{\hat{\mu}_i}{N_i(t)}\frac{dN_i}{dt}. \tag{2}$$

Here, $\hat{\nu}_i$ is an adaptation rate describing the sensitivity to deviations of the actual inventory $N_i(t)$ from the desired one N_i^0. $\hat{\mu}_i$ is a dimensionless parameter reflecting the responsiveness to relative deviations $(dN_i/dt)/N_i(t)$ from the stationary equilibrium state. Moreover, economic systems have an important additional equilibration mechanism, as undesired inventory levels and inventory changes can be compensated for by adjusting the price $P_i(t)$ of commodity i. If the same criteria are applied, we find

$$\frac{1}{P_i(t)}\frac{dP_i}{dt} = \nu_i\left(\frac{N_i^0}{N_i(t)} - 1\right) - \frac{\mu_i}{N_i(t)}\frac{dN_i}{dt}. \tag{3}$$

We will assume that $\alpha_i = \hat{\nu}_i/\nu_i$ is the ratio between the adjustment rate of the output flow and the adjustment rate of the price in sector i. For simplicity, the same ratio $\alpha_i = \hat{\mu}_i/\mu_i$ is assumed for the responsiveness.

Increased prices $P_i(t)$ have a negative impact on the consumption rate $Y_i(t)$ and vice versa, which is described here by a standard demand function f_i with a negative derivative $f_i'(P_i) = df_i(P_i)/dP_i$:

$$Y_i(t) = [Y_i^0 + \xi_i(t)]f_i(P_i(t)). \tag{4}$$

This formula takes into account random fluctuations $\xi_i(t)$ over time around a certain average consumption rate Y_i^0 and assumes that the average value of $f_i(P_i(t))$ is normalized to 1. The fluctuation term $\xi_i(t)$ is introduced here in order to indicate that the variation of the consumption rate is a potentially relevant source of fluctuations. Nevertheless, we do not focus on the investigation of noise-induced effects in this paper as we mainly want to present a model which can reproduce observed features of macroeconomic dynamics *without* having to assume fluctuations to account for unexplained effects (see Sec. IV).

Inserting Eq. (4) into Eq. (1) results in

$$\frac{dN_i(t)}{dt} = Q_i(t) - \sum_{j=1}^{m} a_{ij}Q_j(t) - [Y_i^0 + \xi_i(t)]f_i(P_i(t)). \tag{5}$$

For certain specifications of $f_i(P_i)$ and finite fluctuation amplitudes $\xi_i(t)$ one may therefore find phenomena like stochastic resonance [19]. In our simulations, we have applied the common linear demand function

$$f_i(P_i) = \max(0, f_i^0 - f_i^1 P_i) \tag{6}$$

used by economists, where f_i^0 and f_i^1 are non-negative parameters.

We would also like to note that Eqs. (2) and (3) may be simplified by Cole-Hopf transformations, while it does not help to reduce Eq. (1). Defining $\tilde{N}_i(t) = \ln[N_i(t)/N_i^0]$, $\tilde{P}_i(t) = \ln[P_i(t)/P_i^0]$, and $\tilde{Q}_i(t) = \ln[Q_i(t)/Q_i^0]$, we can write

$$\frac{d\tilde{Q}_i}{dt} = \hat{\nu}_i(e^{-\tilde{N}_i(t)} - 1) - \hat{\mu}_i\frac{d\tilde{N}_i}{dt} \tag{7}$$

and

$$\frac{d\tilde{P}_i}{dt} = \nu_i(e^{-\tilde{N}_i(t)} - 1) - \mu_i\frac{d\tilde{N}_i}{dt}. \tag{8}$$

III. DYNAMIC BEHAVIOR AND LINEAR STABILITY ANALYSIS

The possible dynamic behaviors of supply networks can be studied by analytical investigation of limiting cases (see the Appendix) and by means of a linear stability analysis around the equilibrium state [in which we have $N_i(t) = N_i^0$, $Y_i(t) = Y_i^0$, $Q_i^0 - \Sigma_j a_{ij}Q_j^0 = Y_i^0$, and $P_i(t) = P_i^0$]. Supply systems without a mechanism analogous to price adjustment are covered by cases where $\alpha_i \to \infty$ or $f_i(P_i) = \text{const}$ for all i (no price sensitivity).

Denoting the m eigenvalues of the input matrix **A** by ω_i with $|\omega_i| < 1$, the $3m$ eigenvalues of the linearized model equations are 0 (m times) and

$$\lambda_{i,\pm} \approx \frac{1}{2}(-A_i \pm \sqrt{(A_i)^2 - 4B_i}), \tag{9}$$

where

$$A_i = \mu_i[C_i + \alpha_i D_i(1 - \omega_i)],$$

$$B_i = \nu_i[C_i + \alpha_i D_i(1 - \omega_i)],$$

$$C_i = P_i^0 Y_i^0 |f_i'(P_i^0)|/N_i^0,$$

$$D_i = Q_i^0/N_i^0. \tag{10}$$

Formula (9) becomes exact when the matrix **A** is diagonal or the parameters $\mu_i C_i$, $\alpha_i\mu_i D_i$, $\nu_i C_i$, and $\alpha_i\nu_i D_i$ are sector-independent constants, otherwise the eigenvalues must be numerically determined. It turns out that the dynamic behavior mainly depends on the parameters α_i, ν_i/μ_i^2, and the eigenvalues ω_i of the input matrix **A** (see Fig. 1): In the case $\alpha_i \to 0$ of fast price adjustment, the eigenvalues $\lambda_{i,\pm}$ are given by

$$2\lambda_{i,\pm} = -\mu_i C_i \pm \sqrt{(\mu_i C_i)^2 - 4\nu_i C_i}, \tag{11}$$

i.e., the network structure does not matter at all. We expect an exponential relaxation to the stationary equilibrium for $0 < \nu_i/\mu_i^2 < C_i/4$, otherwise damped oscillations. Therefore immediate price adjustments or similar mechanisms are an efficient way to stabilize economic and other supply systems. However, any delay $(\alpha_i > 0)$ will cause damped or growing oscillations, if complex eigenvalues $\omega_i = \text{Re}(\omega_i) + i\,\text{Im}(\omega_i)$ exist. *Note that this is the normal case, as typical supply networks in natural and man-made systems are characterized by complex eigenvalues* (see the top of Fig. 1). Damped oscillations can be shown to result if all values

$$\nu_i/\mu_i^2 = \alpha_i\hat{\nu}_i/\hat{\mu}_i^2 \tag{12}$$

lie below the instability lines,

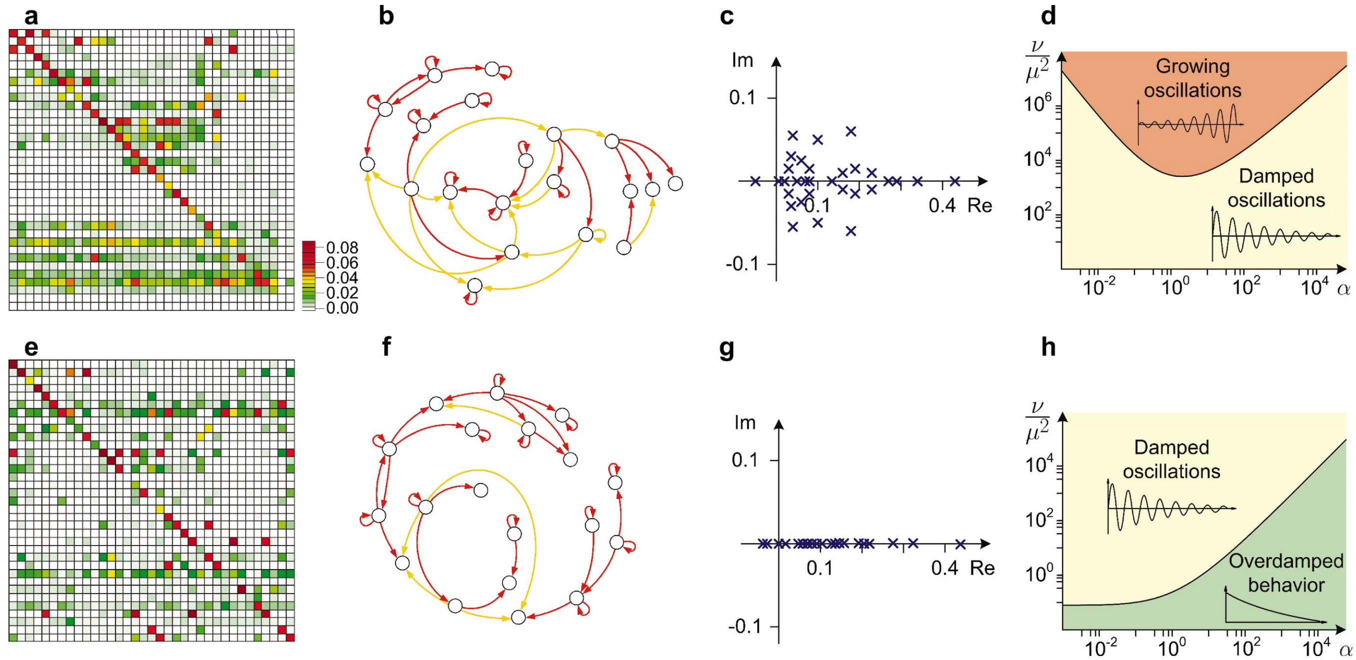

FIG. 1. (Color) Properties of our dynamic model of supply networks for a characteristic input matrix specified as average input matrix of macroeconomic commodity flows of several countries (top) and for a synthetic input matrix generated by random changes of input matrix entries until the number of complex eigenvalues was eventually reduced to zero (bottom). Subfigures (a) and (e) illustrate the color-coded input matrices **A**, (b) and (f) the corresponding network structures, when only the strongest links (commodity flows) are shown, (c) and (g) the eigenvalues $\omega_i = \mathrm{Re}(\omega_i) + i\,\mathrm{Im}(\omega_i)$ of the respective input matrix **A**, and (d) and (h) the phase diagrams indicating the stability behavior of the model equations (1)–(4) on a double-logarithmic scale as a function of the model parameters $\alpha_i = \alpha$ and $\nu_i/\mu_i^2 = \nu/\mu^2 = V/M^2$. The other model parameters were set to $\nu_i = C_i = D_i = P_i^0 = N_i^0 = Y_i^0 = 1$. Surprisingly, for empirical input matrices **A**, one never finds an overdamped, exponential relaxation to the stationary equilibrium state, but network-induced oscillations due to complex eigenvalues ω_i.

$$\nu_i/\mu_i^2 \approx \{C_i + \alpha_i D_i[1 - \mathrm{Re}(\omega_i)]\} \times \left(1 + \frac{\{C_i + \alpha_i D_i[1 - \mathrm{Re}(\omega_i)]\}^2}{[\alpha_i D_i\,\mathrm{Im}(\omega_i)]^2}\right) \tag{13}$$

given by the condition $\mathrm{Re}(\lambda_{i,\pm}) \leq 0$. For identical parameters $\nu_i/\mu_i^2 = \nu/\mu^2$ and $\alpha_i = \alpha$, the minimum of these lines agrees exactly with the numerically obtained curve in Fig. 1(d). Values above this line cause small oscillations to grow over time. Note that a synchronization of the oscillations is not the typical case (although it may occur under certain circumstances [14]).

In some cases, all eigenvalues ω_i of the input matrix **A** are real. This applies to symmetric matrices **A** and matrices equivalent to Jordan normal forms. Hence the existence of loops in supply networks is no sufficient condition for complex eigenvalues ω_i [see also Fig. 1(f)]. It is also no necessary condition. Asymmetric matrices with real eigenvalues belong, for example, to sequential supply chains or directed Cayley trees with equally weighted branches (and some other symmetric distribution networks). In these cases, Eq. (9) predicts a stable, overdamped behavior if all values $\nu_i/\mu_i^2 = \alpha_i \hat{\nu}_i/\hat{\mu}_i^2$ lie below the lines

$$\nu_i/\mu_i^2 \approx [C_i + \alpha_i D_i(1 - \omega_i)]/4 \tag{14}$$

defined by $\min_i(A_i^2 - 4B_i) > 0$. For identical parameters $\nu_i/\mu_i^2 = \nu/\mu^2$ and $\alpha_i = \alpha$, the minimum of these lines corresponds exactly to the numerically determined curve in Fig. 1(h). Above it, one observes damped oscillations around the equilibrium state, but growing oscillations are not possible. In supply systems without a price adjustment or comparable mechanism (i.e., for $\alpha_i \to \infty$ or $C_i = 0$), Eq. (14) predicts an overdamped behavior for real eigenvalues ω_i and

$$\hat{\nu}_i/\hat{\mu}_i^2 < D_i(1 - \omega_i)/4 \tag{15}$$

for all i, while Eq. (13) implies the stability condition

$$\hat{\nu}_i/\hat{\mu}_i^2 < D_i[1 - \mathrm{Re}(\omega_i)]\{1 + [1 - \mathrm{Re}(\omega_i)]^2/\mathrm{Im}(\omega_i)^2\} \tag{16}$$

for all i, given that some eigenvalues ω_i are complex.

IV. EXPLANATION OF SOME EMPIRICALLY OBSERVED FEATURES OF BUSINESS CYCLES

Asynchronous oscillations seem to be characteristic for economic systems. Due to phase shifts between sectors, they imply that the aggregate behavior displays slow variations of small amplitude compared to the single sectors (see Fig. 2). If the function $f_i(P_i)$ and the parameters ν_i/μ_i^2 are suitably specified, the nonlinearities in Eqs. (1)–(4) will additionally limit the oscillation amplitudes, as low inventories $N_i(t) \approx 0$ will cause diverging prices $P_i(t) \to \infty$, which in turn implies vanishing consumption $Y_i(t) = 0$. The resulting equation can be written

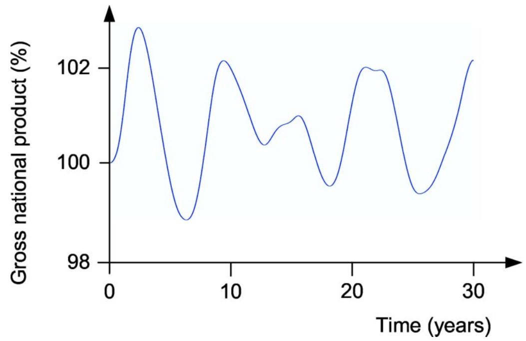

FIG. 2. Typical simulation result of the time-dependent gross domestic product $\Sigma_i Q_i(t) P_i(t)$ in percent, i.e., relative to the initial value. The input matrix was chosen as in Figs. 1(a)–1(d), but Y_i^0 was determined from averaged input-output data. Q_i^0 was obtained from the equilibrium condition, and the fluctuations $\xi_i(t)$ were specified as a Gaussian white noise with mean value 0 and variance $\sigma = 10.000$ (about 10% of the average final consumption). The initial prices $P_i(0)$ were selected from the interval [0.9;1.1]. Moreover, in this example we have assumed $f_i(P_i) = \max[0, 1 + d(P_i - P_i^0)]$ with $d = f'(P_i^0) = -10$ and the parameters $\nu_i = 0.1$, $\mu_i = 0.0001$, $\alpha_i = 1 = P_i^0$, and $N_i^0 = Y_i^0$. Although this implies a growth of small oscillations [cf. Fig. 1(d)], the oscillation amplitudes are rather limited. This is due to the nonlinearity of model equations (1)–(4) and due to the phase shifts between oscillations of different economic sectors *i*. Note that irregular oscillations with frequencies between 4 and 6 yr and amplitudes of about 2.5% are qualitatively compatible with empirical business cycles. Our material flow model can explain *w*-shaped, nonperiodic oscillations without having to assume technological shocks or externally induced perturbations. The long-term growth of national economies was intentionally not included in the model in order to separate this effect from network-induced instability effects.

$$\frac{dN_i(t)}{dt} = Q_i(t) - \sum_{j=1}^{m} a_{ij} Q_j(t), \qquad (17)$$

which implies growing inventories as long as consumption is absent: Just assume that i was the first sector for which the inventory $N_i(t)$ became zero at some point in time t_1. Then, $Q_i(t)$ would diverge at time t_1 and $Q_i(t)$ would dominate $\Sigma_j a_{ij} Q_j(t)$ because of $a_{ij} < 1$, i.e., dN_i/dt would be positive and $N_i(t)$ would not drop below zero.

Our business cycle theory differs from the dominating one [17] in several favorable aspects:

(i) Although finite perturbations may actually occur in economic systems, our theory does not have to assume exogeneous shocks in order to explain business cycles. In the case of growing oscillations they would rather emerge without any external driving, just on the basis of decentralized adjustments in the different sectors of an economic production network. It is, by the way, surprising that increasing oscillation amplitudes are found if the adaptation rates ν_i are large. Nevertheless, many common production strategies suggest to keep constant inventories N_i^0, which potentially destabilizes economic systems. Ideal values of ν_i/μ_i^2 should lie below the instability line (13); see Fig. 1(d). (The stabilizing adjustment to changes dN_i/dt in the inventories is a difficult task, as the time derivatives of empirical measurements are considerably fluctuating quantities. The use of exponentially smoothed data, however, would cause delayed reactions.)

(ii) More importantly, our theory explains irregular, i.e., nonperiodic oscillations in a natural way (see Fig. 2). For example, *w*-shaped oscillations result as superposition of the asynchronous oscillations in the different economic sectors, while other theories have to explain this observation by assuming external perturbations (e.g., due to technological innovations).

(iii) Although our model may be extended by variables such as the labor market, interest rates, etc., we consider it as a potential advantage that we did not have to couple variables in our model which are qualitatively that different. Our model rather focusses on the material flows among different sectors. In this sense, it approaches the problem from a physics point of view. However, it remains to be investigated in the future how successfully our model can forecast the macroeconomic dynamics in this simple form.

V. SUMMARY AND OUTLOOK

While previous studies have focused on the synchronization of oscillators in different network topologies [1,20], we have found that *many supply networks display damped oscillations, even when their units—and linear chains of these units—behave in an overdamped way. Furthermore, networks of damped oscillators tend to produce growing (and mostly asynchronous) oscillations*. Due to the sensitivity of supply systems to their network structure, network theory [1,2,8] can make useful contributions: On the basis of Eqs. (13) and (14) one can design stable, robust, and adaptive supply networks ("network engineering"). For example, it is possible to identify structural and control policies which have a dampening effect. However, in systems with competing goals (such as intersecting traffic streams), oscillatory solutions can be favorable. The results presented in this study and the applied analytical techniques could be also used and generalized to model the dynamics in metabolic networks [11,12] to enhance the robustness of production processes, or to optimize disaster management [15].

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APPENDIX: MATHEMATICAL SUPPLEMENT

1. Linearized model equations

The linear stability analysis is based on the following linearized equations for the deviations: $n_i(t) = N_i(t) - N_i^0$, $p_i(t) = P_i(t) - P_i^0$, and $q_i(t) = Q_i(t) - Q_i^0$, from the equilibrium state:

$$\frac{dn_i}{dt} = q_i - \sum_j a_{ij} q_j - Y_i^0 f_i'(P_i^0) p_i - \xi_i(t), \qquad (A1)$$

$$\frac{dp_i}{dt} = \frac{P_i^0}{N_i^0}\left(-\nu_i n_i - \mu_i \frac{dn_i}{dt}\right), \quad (A2)$$

$$\frac{dq_i}{dt} = \frac{\alpha_i Q_i^0}{N_i^0}\left(-\nu_i n_i - \mu_i \frac{dn_i}{dt}\right). \quad (A3)$$

This system of coupled differential equations describes the response of the inventories, prices, and production rates to variations $\xi_i(t)$ in the demand. The corresponding eigenvalues are shown in Eq. (9).

2. Dynamic behavior in limiting cases

Despite their mathematical similarity, Eqs. (2) and (3) have a surprisingly different impact on the macroeconomic dynamics:

(i) In the case $\alpha_i \to 0$ of fast price adjustment, one can eliminate Eq. (A3) by assuming $q_i(t) \approx 0$ and $Q_i(t) \approx Q_i^0$, so that $dn_i/dt \approx Y_i^0 |f'(P_i^0)| p_i(t) - \xi_i(t)$. Inserting Eq. (A2) into the time-derivative of this equation finally results in the equations

$$\frac{d^2 n_i}{dt^2} + \mu_i C_i \frac{dn_i}{dt} + \nu_i C_i n_i \approx -\frac{d\xi_i}{dt} \quad (A4)$$

of damped harmonic oscillators with eigenfrequencies $\omega_i^0 = \sqrt{\nu_i C_i}$, damping constants $\gamma_i = \mu_i C_i/2$, and external driving $-d\xi_i/dt$ due to variations in the consumption rate.

(ii) In the case $\alpha_i \gg 1$ of slow price adjustment or in supply networks for which a price variable is not relevant, one can eliminate Eq. (A2) by assuming $p_i(t) \approx 0$ and $P_i(t) \approx P_i^0$, so that $dn_i/dt \approx q_i - \sum_j a_{ij} q_j - \xi_i$. Deriving this with respect to time and inserting Eq. (A3) delivers

$$\frac{d^2 n_i}{dt^2} + \sum_j (\delta_{ij} - a_{ij}) \alpha_j D_j \left[\mu_j \frac{dn_j}{dt} + \nu_j n_j(t)\right] \approx -\frac{d\xi_i}{dt}, \quad (A5)$$

where $\delta_{ij} = 1$ for $i = j$, otherwise $\delta_{ij} = 0$. If we assume sector-independent constants $\alpha_i \mu_i D_i = M$ and $\alpha_i \nu_i D_i = V$, the $2m$ eigenvalues $\lambda_{i,\pm}$ are given by

$$2\lambda_{i,\pm} = -M(1-\omega_i) \pm \sqrt{[M(1-\omega_i)]^2 - 4V(1-\omega_i)}. \quad (A6)$$

For empirical input matrixes $\mathbf{A}$, one never finds an overdamped, exponential relaxation to the economic equilibrium, but network-induced oscillations (see Fig. 1). An overdamped behavior is only possible if all eigenvalues ω_i are real numbers.

(iii) If $(A_i)^2/B_i \gg 1$, the eigenvalues become

$$\lambda_{i,-} \approx -A_i \text{ and } \lambda_{i,+} \approx -B_i/A_i = -\nu_i/\mu_i \quad (A7)$$

(where we have used the Taylor expansion $\sqrt{1+\epsilon} \approx 1 + \epsilon/2 + \cdots$). This situation corresponds to a relaxation to the equilibrium state in the case of a large responsiveness $\mu_i \gg 1$. An overdamped behavior is found if all eigenvalues ω_i are real numbers or if all $\alpha_i = 0$, otherwise one expects network-induced oscillations. Interestingly enough, $\mu_i \gg 1$ implies $(dP_i/dt)/(\mu_i P_i) \approx 0$, so that Eq. (3) reduces to $dN_i/dt \approx \nu_i [N_i^0 - N_i(t)]/\mu_i$. Therefore $N_i(t) \approx N_i^0$ and $dQ_i/dt \approx 0$ (i.e., $Q_i \approx Q_i^0$). Inserting this into Eq. (1) yields an implicit equation for the price $P_i(t)$ as a function of the fluctuations $\xi_i(t)$ in the consumption rate, as usually assumed in economics. It reads

$$[Y_i^0 + \xi_i(t)] f_i(P_i(t)) \approx Q_i^0 - \sum_j a_{ij} Q_j^0 = \text{const}. \quad (A8)$$

3. Boundary between damped and growing oscillations

Starting with Eq. (9), stability requires the real parts $\text{Re}(\lambda_i)$ of all eigenvalues λ_i to be nonpositive. Therefore the stability boundary is given by $\max_i \text{Re}(\lambda_i) = 0$. Writing

$$C_i + \alpha_i D_i (1 - \omega_i) = \theta_i + i\beta_i \quad (A9)$$

with $C_i = P_i^0 Y_i^0 |f_i'(P_i^0)| / N_i^0$ and defining

$$\theta_i = C_i + \alpha_i D_i [1 - \text{Re}(\omega_i)],$$

$$\beta_i = \mp \alpha_i D_i \,\text{Im}(\omega_i) \text{ (complex conjugate eigenvalues)},$$

$$\gamma_i = 4\nu_i/\mu_i^2, \quad (A10)$$

we find

$$2\lambda_i/\mu_i = -\theta_i - i\beta_i + \sqrt{R_i + iI_i} \quad (A11)$$

with

$$R_i = \theta_i^2 - \beta_i^2 - \gamma_i \theta_i,$$

$$I_i = 2\theta_i \beta_i - \gamma_i \beta_i. \quad (A12)$$

The real part of Eq. (A11) can be calculated via the relation

$$\text{Re}(\sqrt{R_i \pm iI_i}) = \sqrt{\frac{1}{2}\left(\sqrt{R_i^2 + I_i^2} + R_i\right)}. \quad (A13)$$

The condition $\text{Re}(2\lambda_i/\mu_i) = 0$ is fulfilled by $\gamma_i = 0$ and

$$\gamma_i = 4\theta_i (1 + \theta_i^2/\beta_i^2), \quad (A14)$$

i.e., the stable regime is given by

$$\frac{\gamma_i}{4} = \frac{\nu_i}{\mu_i^2} = \frac{\alpha_i \hat{\nu}_i}{\hat{\mu}_i^2} \leq \theta_i \left(1 + \frac{\theta_i^2}{\beta_i^2}\right) \quad (A15)$$

for all i, corresponding to Eq. (13).

Boundary between damped oscillations and overdamped behavior: For $\alpha_i > 0$, the imaginary parts of all eigenvalues λ_i vanish if $\text{Im}(\omega_i) = 0$ (i.e., $\beta_i = 0$) and if $R_i \geq 0$. This requires

$$\frac{4\nu_i}{\mu_i^2} = \gamma_i \leq \theta_i - \frac{\beta_i^2}{\theta_i} = \theta_i = C_i + \alpha_i D_i (1 - \omega_i) \quad (A16)$$

for all i, corresponding to Eq. (14).

[1] S. H. Strogatz, Nature (London) **410**, 268 (2001).
[2] D. J. Watts and S. H. Strogatz, Nature (London) **393**, 440 (1998); S. H. Strogatz, *ibid.* **410**, 268 (2001); R. Albert and A.-L. Barabási, Rev. Mod. Phys. **74**, 47 (2002); S. Maslov and K. Sneppen, Science **296**, 910 (2002); S. Bornholdt and H. G. Schuster, *Handbook of Graphs and Networks* (Wiley, Weinheim, 2003).
[3] R. Milo, S. Shen-Orr, S. Itzkovitz, N. Kashtan, D. Chklovskii, and U. Alon, Science **298**, 824 (2002).
[4] F. Liljeros, C. R. Edling, L. A. N. Amaral, H. E. Stanley, and Y. Aberg, Nature (London) **411**, 907 (2001); D. J. Watts, P. S. Dodds, and M. E. J. Newman, Science **296**, 1302 (2002).
[5] B. A. Huberman, *The Laws of the Web* (MIT, Cambridge, MA, 2000); R. Albert, H. Jeong, and A.-L. Barabási, Nature (London) **406**, 378 (2000).
[6] U. Alon, M. G. Surette, N. Barkai, and S. Leibler, Nature (London) **397**, 168 (1999).
[7] A.-M. Neutel, J. A. P. Heesterbeek, and P. C. de Ruiter, Science **296**, 1120 (2002).
[8] I. Stewart, Nature (London) **427**, 601 (2004).
[9] S. Koya and T. Ueda, ACH-Models Chem. **135**(3), 297 (1998).
[10] O. N. Bjørnstad, S. M. Sait, N. C. Stenseth, D. J. Thompson, and M. Begon, Nature (London) **409**, 1001 (2001).
[11] S. Dane, P. G. Sørensen, and F. Hynne, Nature (London) **402**, 320 (1999).
[12] M. B. Elowitz and S. Leibler, Nature (London) **403**, 335 (2000); E. Almaas, B. Kovács, T. Vicsek, Z. N. Oltvai, and A.-L. Barabási, *ibid.* **427**, 839 (2004); A. S. Mikhailov and B. Hess, J. Biol. Phys. **28**, 655 (2002).
[13] I. Diaz-Rivera, D. Armbruster, and T. Taylor, Math. Op. Res. **25**, 708 (2000); J. Dejonckheere, S. M. Disney, M. R. Lambrecht, and D. R. Towill, Eur. J. Oper. Res. **147**, 567 (2003); C. Daganzo, *A Theory of Supply Chains* (Springer, New York, 2003).
[14] D. Helbing, New J. Phys. **5**, 90.1 (2003).
[15] D. Helbing and C. Kühnert, Physica A **328**, 584 (2003).
[16] A. Mullineux, D. G. Dickinson, and W. Peng, *Business Cycles* (Blackwell, Oxford, 1993).
[17] J. B. Long, Jr. and C. I. Plosser, J. Polit. Econ. **91**(1), 39 (1983); G. W. Stadler, J. Econom. Lit. **32**(4), 1750 (1994).
[18] W. W. Leontief, *Input-Output-Economics* (Oxford University, New York, 1966).
[19] L. Gammaitoni, P. Hänggi, P. Jung, and F. Marchesoni, Rev. Mod. Phys. **70**, 223 (1998).
[20] L. F. Lago-Fernández, R. Huerta, F. Corbacho, and J. A. Sigüenza, Phys. Rev. Lett. **84**, 2758 (2000).